

CALC BLUE FLIPPED LECTURE PROBLEMS

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These problems are structured for a weekly instantiation with 14 weeks. There are more problems than can reasonably be done in a class period: core [C] and optional [O] problems are listed.

Chapters are indexed by Calculus BLUE series numbering. B1.5 = BLUE Volume 1 Chapter 5, etc.

A few chapters cover “bonus” material and do not have problems.

Most of these problem are outlined/sketched, and do not come with solutions.

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WEEK 1: B1.1-6

Topics: lines, planes, hyperplanes, curves, surfaces, coordinates, vectors, dot products, cross products, scalar triple products.

C1: What is the difference between the implicit formula for lines in 2-D and planes in 3-D? Why doesn't the equation $2x - 3y = 6$ give you a line in 3-D?

Emphasize the dichotomy between implicit and parametrized representation.

One way to define a line implicitly is via two planes, and considering the intersection... Does this give you a way to express a line in 3-D implicitly?

C2: How do you find the intersection of planes $3x + y - z = 4$ and $x - 2y + z = 1$? Can you parametrize the intersection?

Students will try to find a point and will be led to algebraic systems of equations – foreshadowing row reduction and matrices.

C3. Where does the line $x(t) = 2t - 1, y(t) = 3t + 2, z(t) = 4t$ intersect the plane $4x + 3y - z = 3$?

Students will easily see to substitute and solve.

What happens if it's not a plane but a surface?

Is there a unique intersection?

C4: Let's go back to parametrized lines. Now, compute a parametrization of the line between points in \mathbb{R}^4 given by $(1,3,5,6)^T$ and $(2, -1,4,3)^T$.

Use vector language and visualization. Emphasize the role of how language influences thought. Are the two orientations really different with respect to the line? Foreshadow the importance of orientation.

C5: What is the angle between the planes $x - 2y + 3z = 6$ and $2x + 3y - z = 11$?

What is the definition of an angle between planes?

Do the right hand sides matter? Why or why not?

C6: For what value of constant c are the planes $2cx - y + c^2z = 15$ and $x + 5cy - 3z = 4$ orthogonal?

Is there more than one answer?

Is there any answer?

C7: What is the angle between the grand diagonal of a cube and a typical edge?

This is a great problem. Start with 2-d, then 3-d, then n-d. What happens as $n \rightarrow \infty$?

C8: What is the area of the triangle in the plane with vertices at $(1,3)$, $(-2,0)$, and $(5,2)$?

C9: Compute the orthogonal projection of $(8, 2, -5, 6)^T$ onto $(1, -1, 1 - 1)^T$.

C10: Compute the volume of the parallelepiped in 3-d defined by $(1, -2, 3)^T$, $(-3, 4, 0)^T$, $(2, 0, -5)^T$

O1. In how many ways can two lines in \mathbb{R}^3 intersect? How do they usually intersect? What about two planes in \mathbb{R}^3 or \mathbb{R}^4 ?

Lots of discussion. Be sure to discuss that two planes in 4-d intersect in a point.

Discuss the “most likely” type of intersection in terms of bets and then probability.

O2. What is the distance between the following pair of points in \mathbb{R}^4 ? *[ask for coordinates of points]*

Remark on Hamming distance, binary sequences, DNA sequencing, digital signals, etc.

O3. Consider hyperplanes in \mathbb{R}^n for various values of n given by the equation

$$\sum_{i=1}^n \frac{x_i}{C_i} = 1$$

What do these look like? Who cares about such things?

Try to explain how to draw pictures in higher dimensions via cartoons.

O4. Consider the ball of radius $1/2$ inscribed in a unit cube in \mathbb{R}^n . Try to guess its volume as $n \rightarrow \infty$?

Foreshadowing: it limits quickly to zero. Discuss: where is all the volume? In the corners?

Now, inscribe maximal cubes up in the corners outside the original ball. What fraction of volume do they take up as $n \rightarrow \infty$? This leads to a nice logarithmic limit and a final answer of, again, zero.

Good segue to data science.

O5: What is the average length-squared(?) of a random binary vector in \mathbb{R}^n ?

How would you interpret this geometrically?

Get them to think about probability early. Hint at the importance of $O(\sqrt{n})$.

O6: What is the average dot product of random vectors at vertices of cube of side length 2? All entries are ± 1 .

This should get them thinking...

O7: Now, repeat the last problem but restrict to a “positive” region in \mathbb{R}^n , where all entries are either 0 or 1?

Surely, this must have a different answer, right?

O8: When u , v , & w are mutually perpendicular, what can you say about $u \times v \times w$?

WEEK 2: B1.7-12

Topics: vector calculus, velocity/acceleration, curvature, matrices, matrix algebra, equations, and row reduction.

C1: Compute the velocity and acceleration of the curve with components $(\cosh t, \sinh t, \tanh t)$ to review hyperbolic trig functions.

C2: Compute the length of a general helix in 3-d with radius R and height C . What are the asymptotics for small R , C ? Why does this make sense?

C3: Why is the unit normal vector N always perpendicular to T ?

Try to get students to differentiate the equation $T \cdot T = 1$ in order to show that $T \cdot N = 0$. Discover the proof slowly.

C4: Here is a matrix: what is its size? What is its transpose? Decompose this into diagonal, strictly upper, and strictly lower triangular versions.

C5: review n -by- m matrices; ask about which can be multiplied in which orders.

C6: Practice examples of matrix-vector multiplication by linear combinations of columns.

Foreshadowing: this is going to become very important, so learn it now!

C7: Convert the following system to $Ax = b$. [Write out specific examples of linear equations] What is what?

C8: Solve the following:

$$\begin{aligned}x - y + 2z &= 0 \\2x + y - 3z &= 1 \\-3x + 2y + z &= 2\end{aligned}$$

C9: Let c be a constant. For what values of c does this system have a solution?

$$\begin{aligned}x + y + cz &= 1 \\x + cy + z &= 1 \\cx + y + z &= c\end{aligned}$$

C10: Compute the intersection of two planes via row reduction...

Hearken back to Lecture 1...

O1: Compute the length of a product curve given by a curve in the (x, y) plane and third linear component along the z axis.

O2: Compute the length of a product circle in \mathbb{R}^4 given in terms of two planar circles with two different radii. By what factor does the arclength increase?

Lots of discussion about what a circle means in higher dimensions...

How could one generalize this to \mathbb{R}^{2n} ?

O3: Here is a binary friendship matrix for twitter: who is popular? Who is a lurker? Who is a clique?

O4: Here is a cartoon representation of a correlation matrix from neuroscience data. Where is the visual cortex? The motor cortex? What if the neurons are all scrambled? How to disentangle?

O5: Here is an object-attributes matrix of people and medical data. How many questions do you need to ask people in order to figure out who they are? Find first an upper bound, then a lower bound.

O6: What is the derivative of the scalar triple product $r \cdot (v \times a)$?

O7: Is matrix multiplication *associative*? (Review what that means...) Isn't *everything* in life associative?

Come up with counterexamples...Baking anyone?

O8: Can $A^2 = 0$ with $A \neq 0$? What does this mean?

Introduce the concept of a nilpotent matrix.

O9: Let's consider the matrix (promise them it is an important one...):

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

What is A^n ?

Use the binomial theorem and the nilpotent matrix.

WEEK 3: B1.13-18

Topics: inverse, linear transformations, bases, coordinate changes, determinants

C1: Review: here is a **quadratic form**... $Q(x) = x^T Ax$

Why do we call it that?

Plug in variables $x = (u, v)^T$ or $x = (x, y, z)^T$ and see what happens...

C2: Compute the inverse of the following matrices:

2-by-2 random entries.

Have students suggest matrices & compute the inverses.

Compute via the long way, the 3-by-3 matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 2 & 2 \\ -6 & 2 & -6 \\ 3 & -1 & -1 \end{bmatrix}$$

C3: Define block-diagonal matrices: explore what products and inverses (in the square case) look like.

This is an important exercise...

C4: Compute the inverse of a 2-x-2 rotation matrix.

Think! Why does this have to be the answer...

C5: What are the coordinates of $3i - 4j$ in the basis $u = 2i - j$; $v = i + 3j$?

Getting students to convert to an $Ax = b$ problem is critical.

C6: Compute the determinant of a block-diagonal matrix with square diagonal matrices.

C7: Computing the scalar triple product via a determinant.

Remind students of the various symmetries of the scalar triple product & relate to determinant properties.

C8: When are the vectors $(1, 0, 3c)^T$, $(c, 2, -3)^T$, & $(0, 1, c)^T$ coplanar? Colinear?

Get students to think in terms of volumes.

C9: Computing the cross product of two vectors in 3-d via determinants. *[Have students supply entries]*

Does it matter whether we use columns or rows to compute the cross product via a determinant? Why?

O1: Is the inverse of A^2 equal to the square of A^{-1} ?

Do a proof & get the students to fill in details.

O2: Why is $A^0 = I$ for a square matrix A ? Well, assuming $A^p A^q = A^{p+q}$, then...

Fill in the steps of the proof...

O3: Why is the inverse of a product equal to the product of the inverses in reverse order?!?

Compare with transposes...

O4: Definition of an orthogonal matrix, Q .

Show that $Q^{-1} = Q^T$.

O5: Introduce Gram-Schmidt in 2-D.

O6: Is there a good type of "cross product" in \mathbb{R}^n ?

Yes & no... The cross product is ephemeral; the determinant, substantial.

O7: Introduce a Vandermonde determinant & compute it explicitly...

Rows $[1 \ x_i \ x_i^2 \ \dots \ x_i^{n-1}]$ Det = $\prod(x_i - x_j)$.

Do it in 3-D with $x - y - z$.

This is good practice for row-reduction; there are some algebraic subtleties here...

O8: Recall the definition of an orthogonal matrix: $Q^{-1} = Q^T$. What is the determinant? Why?

O9: Recall the geometry of shears: why is the volume unchanged?

Think in terms of integration & volume elements...

O10: Definition of a permutation matrix: each row/column has a unique 1; all else zeros. Tell me about the determinant... the geometry... compositions.

Let students think of properties of these matrices. What might they be good for?

WEEK 4: B2.1-4

Topics: multivariate functions, partial derivatives, derivatives as linear transformations

C1: *Foreshadowing from later in the Volume.* Consider the deflection of a beam:

$$u = \frac{FL^3}{4Ewh^2}$$

Compute all the partial derivatives.

C2: Consider the function which takes the (length, width, height) of a rectangular prism to the pair (volume, surface area); at (2,3,4), which output is most sensitive to changes in which input?

Hang on... does it make sense to compare sensitivities of quantities with different units?

C3: Consider $V = IR$ with $V = 60$, $R = 50$ and V decreasing at 1V/sec, R increasing at 3 Ohm/s. At what rate is the current changing?

Do this via implicit differentiation and then via matrices. Students who had some multi-var calc exposure will want to use implicit diff'n. Ask the class for a more complicated example of a multi-input multi-output function.

C4: Consider the linear function

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 5y \end{pmatrix}$$

Compute all the partials $\partial u/\partial x$, etc. Then, invert the transformation and compute the partials of the inverses. Is $\partial x/\partial u$ equal to the reciprocal of $\partial u/\partial x$? Compare with $u(x) = 3x$.

This foreshadows the Inverse Function Theorem.

C5: Given that

$$[Df] \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

What happens if inputs change at rates $\mathbf{h} = (-2, 2)^T$? *This is doable.*

What if $\mathbf{h} = (3, 3)^T$? *This is not doable, but why not?*

What if you also know that

$$[Df] \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

This will help keep their linear algebra skills fresh.

C6: Consider $PV = nRT$, where n, R are constants. Compute the derivative of P as a function of T and V . Practice computing rates of change.

Again, students will want to use implicit diff'n / chain rule. Emphasize the matrix notation.

O1: Introduce the Cobb-Douglas model for production as a function of Labor & Materials:

$$P = CL^\alpha M^{1-\alpha}$$

Check the doubling property (explaining the nature of the exponents).

If both inputs are increased at unit rate, at what rate does output increase?

O2: What happens if you take the partial derivative of a partial derivative?

Introduce notation for second partials. Foreshadow that mixed second partial derivatives commute. Foreshadow harmonic functions & Laplacians.

O5: What is the derivative of $f(A) = A^3$ or $f(A) = A^{-1}$, where A is a square matrix.

This is intimidating, but it emphasizes the definition of the derivative, and it gets at the Taylor expansion approach. The cube function is easy – just the Binomial Theorem. The inverse is harder: use the geometric series to get the 1st-order term as $-A^{-1}HA^{-1}$. Why does this answer make sense?

WEEK 5 : B2.5-8

Topics: the Chain Rule, differentiation rules, the Inverse and Implicit Function Theorems

C7: Make up a Chain Rule problem with f and g each being polynomial functions, one with 3 inputs / 2 outputs, and the other reversed. Ask which compositions make sense and what the sizes of the derivatives are in each case.

C8: Consider functions f, g , each with 2 inputs / 2 outputs, taking the origin to the origin. Given numerical values for $[Df]$ and $[D(f \circ g)]$ (at the origin), compute $[Dg]$.

Are there any circumstances under which $[Dg]$ might not exist even though the other terms do? Hmmmmmm...

When it comes to the Implicit Function Theorem, students will be intimidated:

review the statement that $\left[\frac{\partial y}{\partial x}\right] = \left[\frac{\partial F}{\partial y}\right]^{-1} \left[\frac{\partial F}{\partial x}\right]$

P1: Let's practice the 1-d case. Can we solve for $y = y(x)$ given that

$$x e^y - y e^x = 1.$$

P2: What is the relationship between the Inverse and Implicit Function Theorems? Solve for $\mathbf{u} = f^{-1}(\mathbf{v})$ given that $\mathbf{v} - f(\mathbf{u}) = \mathbf{0}$. Let $\mathbf{y} = \mathbf{u}$ and $\mathbf{x} = \mathbf{v}$ and apply the Implicit Function Theorem.

P3: A fully nonlinear problem: can you solve for c, d as a function of a, b , given that

$$\begin{aligned} ab - bc + cd &= 8 \\ a + 2b - 2c + 4d &= 12 \end{aligned}$$

Assuming that you are near $a = 1, b = 2, c = 3, d = 4$.

What does this problem mean geometrically?

What does the solution to these equations look like in 4-D?

P4: Now, do it in the linear case... assuming that you are near $a = 0, b = 1, c = -2, d = 2$ solve for c, d if

$$\begin{aligned} a + 2b - 3c + 4d &= 16 \\ -2a + b + c - 3d &= -7 \end{aligned}$$

Do this via the IFT, then do it explicitly using row reduction. Ahha! Row reduction and back-substitution is really the linear version of what the IFT is doing...

O3: What is the derivative of the square Q^2 of the quadratic form $Q(\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x}$?

The derivative of Q was computed in the videolectures.

O4: Infer the old calculus addition rule for derivatives via the chain rule: $(f + g)' = f' + g'$.

Is there any point in the proof in which the rule is already assumed?

Now do the generalized power rule $u(x)^{v(x)}$ via the Chain Rule.

Students will not remember this and it will be impressive when they can derive it easily.

O5: In 3-D, we need to communicate with 4 satellites to make GPS work. How does the number of satellites change if you find yourself on a planet in 2-d? In 4-d?

This should generate some interesting discussions.

WEEK 6 : B2.9-13

Topics: gradients, tangent planes, linearization, differentials, Taylor series, expansion

Review: level sets and the relationship $\nabla f \cdot \mathbf{u} = [Df]\mathbf{u}$

Review: relative rates in terms of derivative of a natural log.

C1: Compute the gradient of $f(x, y) = ax^2 + by^2$ for various constants a, b . What do the level sets look like?

Students need lots of practice conceiving level sets.

C2: Compute the tangent plane to $xyz - 2xy^3 + 3z^2 = 0$ at the point $(3, 1, 1)$.

C3: Compute a parametrized tangent line to $\gamma(t) = (t^2, -3t, t^3)$ at $t = 2$.

C4: Compute a parametrized tangent plane to $S(t_1, t_2) = (t_1 + 3t_2, t_1 t_2, 2t_1^2 - t_2^3)$ at $(2, -1)$.

O1: Consider the area of a parallelogram defined by two vectors in the plane: $A = ab - cd$. Assume all quantities positive. If each of a, b, c, d can vary by 10%, then by what percentage can the area vary? Why is there a power law here?

Students will find the algebra daunting, but it works "miraculously" in the end. Why?

C5: At what points are the level sets of $x^2 + y^2 - 2xy$ and $2y - 3x$ orthogonal?

O2: Is surface area or volume more "sensitive" to errors in 3-d? Assume a 1% error in measurements of length scale. $A = C_A L^2$; $V = C_V L^3$

O3: Find the equation of a tangent hyperplane to a unit sphere in \mathbb{R}^n at a given point \mathbf{x} .

This is conceptually though not algebraically difficult.

C6: Given the multi-index $I = (1, 3, 3, 2)$, what is its degree, $|I|$? What is the factorial $I!$? If $\mathbf{x} = (x_1, x_2, x_3, x_4)$, what is the monomial term \mathbf{x}^I ?

C7: Use the "long way" (i.e., computing lots of partial derivatives) to compute the Taylor series of

$$f(x, y) = 3 - x + 2y + 5xy - y^2$$

about the point $x = 1, y = 2$. Write your answer as a polynomial in the variables $(x - 1)$ and $(y - 2)$.

When done, check your work by multiplying everything out and simplifying back to a regular polynomial.

This is rather tedious.

C8: Give the Taylor expansion about $x = y = 0$ of the following function up to and including terms of order 4.

$$f(x, y) = \ln(1 + x \cos(y - xe^y))$$

Maybe only up to order 3: this is not so easy... Using the chain rule here is clearly the right way to go.

WEEK 7 : B2.14-19

Topics: optimization, applications, constraints, the Lagrange method

C1: Find and classify the critical points of the following: no constraints...

$$f(x, y) = y^4 - 2xy^2 + x^3 - x$$

C2: Find and classify the critical points of the following: no constraints...

$$f(x, y) = e^{-y}(x^2 - y^2)$$

C3: Compute the second derivative matrix ["Hessian"] of the following function of six(!) variables:

$$f(x, y, z, u, v, w) = u^3 - 3uv^2 + v^4 + w^3 - 3wx^2 + x^4 + y^3 - 3yz^2 + z^4$$

This is a good problem, as students have trouble conceptualizing with this many variables.

C4: A metal plate is heated so that the temperature is

$$T(x, y) = 2x^2 + y^2 - y + 3$$

The plate is a disc of the form $D = \{x^2 + y^2 \leq 1\}$. What are the maximal and minimal temperatures on the plate? *That's not so easy, since you have a boundary. You first look for critical points in the interior of the plate, and classify those. Good. NOW, the involved part. To determine optima on the boundary, you can do the following "hack": parameterize the boundary of D as the curve $(x(t), y(t)) = (\cos t, \sin t)$, for $t \in [0, 2\pi]$. Now, plug these into $T(x, y)$ above to get a function $T(t)$: find the max and min temps here, as a single-variable calc-1-style problem.*

C5: Consider the cost function $f = x^2 + y^2$ and the constraint function $g = xy$. Draw pictures of the levels sets of both functions and solve for maxima and minima graphically. Interpret the Lagrange multiplier in terms of rates of change of the optimal value.

Students will find this challenging.

C6: Use a Lagrange multiplier to find the critical points of the function $f(x, y, z) = x^2 + y^2 + z^2$ on the plane $ax + by + cz = 1$.

What is the geometry of this problem? Can you classify any critical points you found? How could you extend this problem from 3-D to n -D?

C7: Use a Lagrange multiplier to re-derive the formula for the minimal distance from a point (x_0, y_0) in the plane to the line $ax + by = c$. Hint: minimize the square of the distance, using $f(x, y) = (x - x_0)^2 + (y - y_0)^2$ subject to the constraint of being on the line. Then, after finding this minimal distance-squared, take its square root.

Can you see how not-so-hard it is to do this in arbitrary dimensions?

O1: For what value of C will the function $f(x, y) = Cx^2 + 4xy + Cy^2$ have a local max at $(0,0)$? Min? Saddle? Classify as a function of C .

Review the applications to linear regression & Nash equilibria.

Introduce GANs = Generative Adversarial Networks & relate to Nash equilibria.

O2: Compute the Nash equilibrium for the following 2-player game, where, recall, the convention is for player A to receive from player B the amount in the corresponding entry of the matrix.

$$P = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$$

Compute the expected payout at the Nash equilibrium. Is there an advantage to being player A or B? In practice, it's clear that the first player has an advantage. What is not clear is how much an advantage it is and with what probability the strategies should be played.

O3: Repeat the analysis from in class of finding a Nash equilibrium in the following 2-player game of "rock-paper-scissors" where the payoff matrix is (unlike the traditional game) non-binary and non-symmetric:

$$P = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 4 & 0 & -3 \end{bmatrix}$$

For this problem, player A will choose strategies with a probability distribution $(a, b, 1 - a - b)^T$ and player B with $(c, d, 1 - c - d)^T$. The average payoff function will hopefully have a single equilibrium.

Finding the equilibrium is an algebraic challenge, but it uses 2-by-2 inverses: that's nice. The resulting answer is quite a surprise.

O4: Observe that the following has a critical point at the origin:

$$f(x, y, z) = 5x^2 + 5y^2 + 9z^2 - 2xz - 2yz$$

Since it has three variables, you can't use the 2-d method we covered in class to classify it, so what do you do?

There are a few options:

You can learn about eigenvalues (do! next semester...) and use them to classify.

You can memorize a complicated rubric called "Sylvester's criterion". Don't: life's too short.

Get on wolframalpha and play with it. That helps in this case.

"Completing the square" -- remember that? Ah, yes... Note that f can be factored into a sum of squares. First, work with x and z :

$$f(x, y, z) = \left(5x^2 - 2xz + \frac{1}{5}z^2\right) + \frac{44}{5}z^2 - 2yz + 5y^2 = \left(\sqrt{5}x - \frac{z}{\sqrt{5}}\right)^2 + \frac{44}{5}z^2 - 2yz + 5y^2$$

Keep going...

In the end, you have expressed f near the origin as a sum of squares, with positive coefficients in front of each square term: thus you have a minimum. This method works in general for a Hessian quadratic form near a critical point. It's not fun, but it does work!

O5: Recall the Cobb-Douglas model of production in economics. *[If did not cover it previously...]* It says that the amount of product P produced depends on the amount of labor x and materials y via:

$$P = \kappa x^\alpha y^\beta$$

where $\alpha + \beta = 1$. Assume that labor costs A dollars per unit, and materials cost B dollars per unit; use a Lagrange multiplier to determine how you should allocate a fixed amount of C dollars so as to maximize production.

Does your answer make sense? How do things change as worker costs increase? If there were a commodity shock of rapidly increasing prices for raw materials, how would it impact unemployment? What would you do to model two classes of labor (workers/management) and different types of raw materials?

O6: Use two Lagrange multipliers to solve the following: what points on the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $2x + 2y + z = 2$ are closest and farthest from the origin?

Hint: as before, you might find it helpful to maximize/minimize the square of the distance to the origin, and then take the square root of the resulting figures. RECALL from what we talked about at the end of class, that for two constraints $G_1 = 0$ and $G_2 = 0$, the Lagrange equations take the form:

$$[Df] = \lambda_1[DG_1] + \lambda_2[DG_2]$$

These, together with the two constraints, allow you to solve for critical point(s).

WEEK 8 : B3.1-5

Topics: integration, definitions and computations, double/triple integrals, averages

C1: Compute the integral over a rectangle:

$$\int_{\pi/6}^{\pi/3} \int_0^{\pi/2} \cos(x - y) \, dy \, dx$$

C2: Compute the integral over the n -dimensional cube:

$$\int_0^1 \int_0^1 \cdots \int_0^1 x_1 \, dx_n dx_{n-1} \cdots dx_2 dx_1$$

Do it in two different orders to see what is going on with Fubini...

Does the answer make sense? Foreshadowing of centroids...

C3: Reverse the order of integration of

$$\int_0^4 \int_{\sqrt{u}}^2 u^2 + v^2 \, dv \, du$$

C4: Reverse the order of integration of

$$\int_0^1 \int_0^{2x} f(x, y) \, dy \, dx$$

C5: Reverse the order of integration of

$$\int_{x=0}^2 \int_{y=0}^{x^2} x^2 - y^2 \, dy \, dx$$

C6: Compute the following triple integral:

$$\int_{x=0}^2 \int_{y=0}^x \int_{z=x}^y 3xyz \, dz \, dy \, dx$$

Why is the answer negative, when the integrand is everywhere positive!? Have we made a mistake? Emphasize orientation and the single-variable case of reversing limits.

C7: Compute the centroid of the upper hemisphere of radius 2.
Maybe don't actually do the integrals, since they are unpleasant.
Be sure to emphasize symmetry arguments & foreshadow the use of polar/spherical coordinates.

C8: Reorder the limits:

$$\int_{y=0}^1 \int_{x=0}^2 \int_{z=0}^{x^2} f \, dz \, dx \, dy = \int_{z=?}^? \int_{y=?}^? \int_{x=?}^? f \, dx \, dy \, dz$$

O1: Compute the value of the improper integral

$$\int_0^\infty \int_0^\infty e^{-ax-by} \, dy \, dx$$

Emphasize the use of Fubini & reduce to two cases of a single integral.
Can you see how easy it would be to do this in n-D?

O2: What does it mean to compute the integral of f over a single point?
Good discussion. Is it always zero? What is it is the integral of $f: \mathbb{R}^0 \rightarrow \mathbb{R}^1$?

O3: Compute the centroid of a tetrahedron in positive orthant bounded by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(for $a, b, c \geq 0$).

Set up the integrals for one coordinate, and then give them the answer (unless there is time to do the full integral).
Then, argue via symmetry to get the other coordinates.

Does this answer make sense? Use a physics argument [point-masses at the corners] as a comparison. Is that a rigorous computation?

WEEK 9 : B3.7-8, 10-11 [9, 12 opt]

Topics: moments of inertia, inertia matrices [opt], solid body mechanics [opt], probability, covariance matrices [opt]

C1: Here are some facts: for spherical balls (solid) and spherical shells (hollow) of radius R and total mass M , the moments of inertia are

$$I_b = \frac{2}{5}MR^2 \quad \text{versus} \quad I_s = \frac{2}{3}MR^2$$

How would you in practice compute these integrals? Set them up...

Of course, we don't want to do those integrals, but we will some day soon...

Is it ever harder to rotate the solid over the shell? Can we even compare? Are the units different?

C2: Compute the moment of inertia of a solid cube rotated through the centroid with density $\rho = x^2 + y^2 + z^2$.

C3: If you know that the disc of radius R in the plane has polar inertia (rotated about the center point) $I_0 = \frac{MR^2}{2}$, then what else can you do? Can you compute the moment of inertia of a cylindrical shell? A solid cylinder? A solid cone?

C4: Consider a solid uniform-density cylinder of radius r and height h , arranged so that the centroid is at the origin and the height is aligned with the z -axis. Now imagine that you slice off the top of this object with a tilted plane

[draw figure]. Which coordinates of the centroid of the remaining solid object **must** have changed?
This is a good visualization challenge.

The probability material is particularly difficult for many students.

C5: Recall the exponential probability density: $\rho = \alpha e^{-\alpha x}$ on $\{x \geq 0\}$. What is its expectation and variance?

C6: Say $D = \{0 \leq x \leq 3, 0 \leq y \leq 1\}$ and $\rho = C(x^2 + 2y^2)$.

What value of C makes this a probability density?

What is the probability that $x \leq y$?

What is the probability that $x < y$?

What are the integrals to compute \mathbb{E} and $[\mathbb{V}]$?

C7: What is the value of C for which $\rho = Cxy$ is a probability density on $0 \leq x, y \leq a$?

O1: Compute the moment of inertia of a unit density solid cube of side length s in \mathbb{R}^n through the centroid (using an axis-aligned rotation axis, of course).

This is good practice at figuring out the r^2 term...

O2: Set up the integrals to compute the inertia matrix $[I]$ for a tetrahedron (from last week) rotated about the x, y, z axes through the origin.

What would you change in order to get the inertia matrix for rotating about the centroid?

O3: For an asymmetric rectangular prism, which axis maximizes/minimizes the moment of inertia?

Perhaps foreshadow the Intermediate Axis Theorem by rotating and seeing what happens...

O4: For a rectangular prism of side lengths a, b, c , what is the angular momentum if the angular velocity is $\omega = (a, b, c)^T$?

O5: Compute the 2-by-2 case of an inertia matrix in 2-D for a rectangular plate of width w and height h . Then use this to rotate it about the diagonal.

Can you get this from the 3-d case? Oh yes...

O6: What is the covariance matrix for the sums of three independent random variables (X, Y, Z) into combinations $(X + Y, Y + Z, X + Z)$? Recall that $[\mathbb{V}(AX)] = A[\mathbb{V}]A^T$

What about higher dimensions?

O7: Consider the joint independent density: $\prod_i \alpha_i e^{-\alpha_i x_i}$.

Compute a marginalization. (*Not so bad...*)

If this models wait-times, what is the probability that x_1 will go first?

This is very challenging...

It should generalize the result of 2 persons from the lecture, which was

$$\frac{\alpha_1}{\alpha_1 + \alpha_2}$$

What about the expectation? Still $\max_i \alpha_i$

Variances: this is a 1-d problem via parts: $\mathbb{V}(X_i) = \alpha_i^{-2}$

WEEK 10 : B3.13-17 [18-19 opt]

Topics: cylindrical & spherical coordinates; coordinate changes; surface integrals; Gaussians & data [opt]

C1: What is the centroid of a solid hemisphere?

Have a discussion on the merits of using cylindrical & spherical. Which is easiest?

C2: What is the moment of inertia of a unit-density solid ball of radius R about the centroid.
What about a spherical shell of the same mass and radius?

C3: A standard Gaussian in 2-d is a product of two standard Gaussians in 1-d. What is the probability of being with 1-, 2-, and 3-standard deviations (assuming unit variance for simplicity)?
This will require a bit of numerical computation on a calculator or computer. Pairs with the bonus material on Gaussians.

C4: Compute the volume of an ellipsoid with semiaxes a, b, c by using a linear change of variables.
This is more difficult than it sounds – one has to be careful with the ordering of things.

C5: Transform the following integral

$$\iint x^2 y^2 (y^2 - x^2) dx dy$$

over the domain given by $1 \leq xy \leq 4$ and $1 \leq y - x \leq 3$.

C6: What coordinate change would you use to evaluate

$$\iint xy (x^2 + y^2) dx dy$$

over the domain given by $1 \leq xy \leq 4$ and $1 \leq x^2 - y^2 \leq 3$.

C7: What is the total charge on sphere when surface charge density is of the form κz^2 ?

C8: Integrate $z(x^2 + y^2)$ over the surface parametrized by $x = u \cos v$; $y = u \sin v$; $z = u$; $0 \leq u, v \leq 1$.

O1: Compute the inertia matrix $[I]$ of a unit-density solid hemisphere of radius R about the center of the sphere. Use the parallel axis theorem to do it about the centroid.

This one gets interesting: is I_{xx} the same as I_{zz} ? Why or why not? Use this to show the efficacy of proper choice of coordinate frame...

Can you solve this by knowing the moment of inertia of the solid ball and cutting it into two halves?

Use this to emphasize additivity of integrals.

O2: If you have the coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (3 + r \cos \phi) \cos \psi \\ (3 + r \cos \phi) \sin \psi \\ r \sin \phi \end{pmatrix}$$

What is the volume element?

This is quite a challenge algebraically.

O3: In the previous problem, can you compute a surface element for a double integral where the r term above is a constant?

O4: We previously investigated what happens when you integrate a 2-d Gaussian of the form $e^{-(x^2+y^2)/2}$ over a disc. Now what about over an ellipse when the variances are different? Say $e^{-(ax^2+by^2)/2}$.

Probably skip this unless bonus material on Gaussians and covariance matrices was covered.

WEEK 11 : B4. 1-5

Topics: Fields, scalar path integrals, 1-forms and 1-form integrals, independence of path, work, circulation, flux

C1: Draw linear vector fields of the form $\vec{F} = \lambda_1 x \hat{i} + \lambda_2 y \hat{j}$ and $\vec{F} = \lambda_1 y \hat{i} + \lambda_2 x \hat{j}$ for various values of constants.

C2: Derive a general formula for parametrizing a straight-line path between two points in \mathbb{R}^2 .

How do things change in higher dimensions?

C3: Integrate the scalar field $f = e^{-x^2-y^2}$ over the circle of radius R about the origin.

What value of R maximizes this integral?

This leads to some interesting discussions: can you maximize without actually doing all the work? Foreshadows Green's theorem.

C4: Compute the integral of $y^2 dx$ over the graph of $y = x^2$ as $-1 \leq x \leq 1$.

What about straight path between endpoints?

This emphasized path-dependence.

C5: Integrate the 1-form field $x dy$ over a rectangular region in the plane.

This foreshadows what comes next.

C6: Investigate the gradient 1-form field $\alpha = df$, where $f = x^2 + y^2$.

Try a straight-line path... a circular path...

C7: Is the 1-form field $x dy - y dx$ the gradient of some function f ?

How can you tell? Try lots of possibilities...

C8: Is the 1-form field $\alpha = e^x \cos y dx - e^x \sin y dy + z dz$ a gradient? Why or why not?

C9: Compute the work W done by the vector field $\vec{F} = xy \hat{i} + yz \hat{j} + xz \hat{k}$ along the straight-line path from $(1,2,0)$ to $(4,3,-1)$.

O1: can you integrate the 1-form field on \mathbb{R}^n given by $\sum_i x_{i+1} dx_i$ (cyclic ordering) over the straight path from $\underline{0}$ to $\underline{1}$? What does this even mean? What about along a different path?

O2: Consider the vector field $\vec{F} = x^2 \hat{i} - 2y \hat{j}$. Is the integral of $\alpha_{\vec{F}}$ positive, negative, or zero over the following paths? *[make up some paths and think/check as discussion leads...]*

O3: What is an example of a scalar field whose integral would be non-infinite on the hyperbola $xy = 1$ where $x, y \geq 0$?

O4: Compute the integral

$$\int_{\gamma} x^2 dx + yz dy + \frac{1}{2} y^2 dz$$

along the path from the origin to $(0,0,10)$ given by γ with $x(t) = e^t \sin 4\pi t$; $y(t) = t(t-1) \cos^2 t$, and $z(t) = 10t^{10}$ for $t = 0 \dots 1$.

Think! Try to find an easier way to do this...

WEEK 12 : B4. 6-8

Topics: Green's Theorem, curl, divergence, 3-D differential forms

C1: What is the flux 1-form of the vector field $\vec{F} = (x^2 + 4y)\hat{i} + (x + y^2)\hat{j}$?

Use this to compute the flux of this field over the square in the plane with corners $(-1,2)$ and $(2,5)$.

Try doing it without Green's Theorem and then with it...

Beware of orientations! Which way is easier?

C2: Compute the circulation [work] done by $\vec{F} = (y + x \sin x^2)\hat{i} + (x^2 + e^{y^2-5y})\hat{j}$ along the circle of radius 2 centered at the origin and oriented counterclockwise.

Just kidding...that's impossible. No wait! It's not! Try using Green's Theorem...

C3: Compute the circulation of a fluid with velocity field $V = (xy + y^2)\hat{i} + (x - y)\hat{j}$ along the [counterclockwise] curve bounded by the graphs of $y = x^2$ and $x = y^2$.

This would require two path integrals, if you solved it with a path integral (wink wink).

The material on div/curl and differential forms is very challenging for students at first.

C4: What is the smart way to remember the signs of the terms in Green's Theorem?

Get students to think in terms of the forms version. Will get lots of questions about the difference between the oriented area 2-form $dx \wedge dy = -dy \wedge dx$ and the area element $dA = dx dy = dy dx$.

C5: Consider the planar vector field $\mathbf{F} = (ax + by)\hat{i} + (ay - bx)\hat{j}$.

What are the curl and divergence of this field?

What does it look like?

What does it mean? Try drawing pictures...

C6: Now make this a 3-D vector field by adding $+cz\hat{k}$. How does this change the divergence? The curl?

C7: Generate simple practice problems on the derivatives of 1-form fields.

C8: Generate simple practice problems on the wedge products of 1-form fields.

O1: Use Green's Theorem to show that the centroid of a region $D \subset \mathbb{R}^2$ is given by the path integrals

$$\bar{x} = \frac{1}{2A} \int_{\partial D} x^2 dy \quad \bar{y} = \frac{1}{2A} \int_{\partial D} y^2 dx$$

where A is the area of D .

Students find this challenging. Write out the formulae for the centroid coordinates as double integrals; then use Green's Theorem to convert the putative right hand side path integrals.

O2: Draw a really complicated, multiple connected region in the plane and ask for the orientations of various boundary components. This can lead to some interesting discussions about orientation.

WEEK 13 : B4.9-11 [+12]

Topics: Gauss' and Stokes' Theorems

C2: What is the integral of the 2-form $\beta = z dx \wedge dy - x^2 dy \wedge dz$ over the surface given by $z = 4 - x^2 - y^2$ with $z \geq 0$.

C3: What is the flux of $\mathbf{F} = x\hat{i} + y^2\hat{j} + (z + y)\hat{k}$ along the boundary of the cylindrical solid within $x^2 + y^2 \leq 4$, below $z = 8$, and above $z = x$. Use an outward-pointing normal.

C4: What is the flux of the field $\mathbf{F} = y^2\hat{i} - x^2\hat{j} + z^2\hat{k}$ across the upper-hemisphere of radius R at the origin? Orient it with the positive z -axis.

C5: What is the circulation of the field $(e^z + 3y \cos x)\hat{i} + (3 \sin x)\hat{j} + (2 + x e^z)\hat{k}$ along the loop given by $x = (5 + \cos 3t) \cos t$; $y = (5 + \cos 3t) \sin t$; $z = \sin t$; $0 \leq t \leq 2\pi$.

C6: Compute the circulation of the field $F = e^{-x}\hat{i} + e^x\hat{j} + e^z\hat{k}$ along the triangle in the first octant cut out by the plane $2x + y + 2z = 2$.

C7: What is the flux of the field $F = (\cos z + xy^2)\hat{i} + (xe^{-z})\hat{j} + (\sin y + x^2z)\hat{k}$ out of the surface given by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. What if you don't have that top disc?

O1: Compute the flux of the vector field $\vec{F} = (y + x \sin x^2)\hat{i} + (x^2 + e^{y^2-5y})\hat{j}$ (in 3-D) through the unit sphere, oriented outwards. Ouch! Maybe a cube of side length two centered at the origin? Also, ouch!

O2: Archimedes' tub! What is the buoyant force on a submerged body?

Discuss the force as the pressure applied to the normal vector to the body surface. Assuming the water is of constant density, the magnitude of the pressure force in the z-direction is ρz times the projected area in the $x - y$ plane.

The big punchline: the buoyant force field is really a 2-form field $\beta = \rho z dx \wedge dy$.

Now, what is the net buoyant force?

This can be done with a physical demo & makes a great application of the divergence theorem...

WEEK 14 : B4.13-18 [BONUS]

Review Green/Gauss/Stokes and do review of the semester for the final exam...